A Sine Fuzzy information Measure with their Applications in MADM Problems and Pattern Recongnition

Vikrant Narwal*, Gurdas Ram[1] and
Satish Kumar[2]
[1,2] Department of Mathematics, Markandeshwar (Deemed to be University), Mullana, Ambala, India
vikrantnarwal02@gmail.com, gurdasdadwal@rediffmail.com, drsatish74@rediffmail.com

Abstract. In the present communication, based on the Sine function, two effective measure are proposed in fuzzy theory, some interesting properties of these measures are analysed. Numerical example is given to show that the information measure of the proposed fuzzy entropy are effective by the comparison of the proposed entropy and the existing entropy. Further, we propose a divergence measure based on Jensen-Sine function which is known as Jensen-Sine divergence measure. It is generalization of J-divergence measure. One of the salient features of this measure is that we can allot the equal weight to each fuzzy set. This makes it specially reasonable for the study of decision making problem. Further, the idea has been generalised from probabilistic to fuzzy divergence measure. At last, the application of proposed Jensen-Sine fuzzy divergence measure is given in strategic decision-making and pattern recognition.
Keywords: Sine fuzzy information measure, Jensen-sine divergence measure, MADM, Pattern Recongnition.

1 Introduction

Entropy is a very important notion for measuring uncertain information. Fuzziness, a feature of imperfect information, results from the lack of crisp distinction between the elements belonging and not belonging to a set. Zadeh[25] was the first person who introduced the notion of fuzzy sets. With the help of theory of fuzzy sets, we can find the vagueness linked with decision making issues. The concept of probability was the only method to find vagueness linked with decision making issues before the introduction of theory of fuzzy sets. In contrast to the traditional mathematical binary representation, FSs gives a powerful picture of truth. The additive nature of membership in FSs makes the principle helpful for displaying the minimal accuracy of mental representations. Zadeh [25] developed linguistic hedges, fuzzy similarity relations, and fuzzy decision-making between 1965 and 1975, widening the basis of FS theory. In the 1970s, several Japanese research groups started to interrogate/examine FS theory. Mamdani discovered the first fuzzy logic controller in 1970. In Europe and Japan, some industrial applications/executions of fuzzy logic started in 1977. The fame of fuzzy logic in Japan in the early 1980s guided to rejuvenation of fuzzy logic in the United States by the end of the decade. In each scientific field, FSs have provided remarkable growth. From engineering to the social sciences and humanities, from information technology to medical sciences, and from biological sciences to physical sciences, it established numerous application areas in conceptual as well as applied studies. After the introduction of FSs, some outstanding studies have been concluded regarding multiple attribute decision-making (MADM) problems under unknown environments. In an FS, an element is assigned a membership value in [0; 1] and the non-membership degree is by default taken as one minus the membership degree so that their sum should come out to be one. This means that the non-membership degree of an element cannot be allocated independently in an FS. So, realizing this, Atanassov [2] proposed the concept of the intuitionistic fuzzy set (IFS) by including the degree of non-membership of an element to the FS with the sum of these membership degrees less or equal to one. Many researchers have paid due attention to IFSs and resolved decision-making problems in the intuitionistic fuzzy (IF) environment with more vigorous results. Standard FSs have been generalised to new variants in recent years, and these forms are commonly used in a variety of fields, including medicine, energy, economics, materials, and pharmacology. The objectives of this paper are: (a) To introduce some generalized versions of fuzzy entropy measure (FEM) that allows the exibility of applications in certain real-life problems. (b) We also consider theoretical and practical justification of the proposed measures by contrasting their performance with various existing fuzzy information measures. (c) The newly proposed measures deal certain intrinsic/extrinsic aspects of fuzziness more efficiently.

In view of the proposed objectives the main contribution of this paper is as follows.

1. We introduce a generalized (parametric) fuzzy entropy measure (GFEM) and compare the practical usability of the proposed GFEM with some existing fuzzy entropy measures (FEMs) and FKMs.
2. We introduce a generalized fuzzy divergence measure (GFDM).
3. We illustrate the implication of the GFEM to MADM. The performance of the proposed fuzzy measures has also been compared with some of the existing fuzzy information measures in context of MADM.

This content of the paper is presented as follows: In Section 2, we present some basic definitions regarding FS theory and fuzzy information measures. In Section 3, we introduce a new information measure and discuss its properties, further it is generalized into fuzzy environment and describe its advantages and justification from various viewpoints. In section 5, new fuzzy divergence measure is introduced and its properties are discussed along with its application in pattern recognition. Section 6 is devoted to the applications of the proposed divergence measure. Finally, Section 7 concludes the paper.

2 Preliminaries

Definition 2.1. Let $\Delta_m = \{Q = q_1, q_2, ...q_m) : q_i \geq 0, i = 1, 2, ...m; \}, m \geq 2$ be set of m-completely probalitic distributions. For any probalitic distribution $Q = (q_1, q_2, ...q_m) \in \Delta_m$, Shannon [3] (1948) defined an entropy as :

$$V(Q) = -\sum_{i=1}^{m} (q_i) \log(q_i). \qquad (1)$$

In this whole paper it is assumed that (0)log(0) = 0 and logarithms to the base D (D > 1).

Tsalli [37] (1988) proposed an entropy as:

$$^{\alpha}V(Q) = \frac{1}{1-\alpha}\left[1-\sum_{i=1}^{m} q_i^{\alpha}\right] \alpha > 0,\ (\neq 1) \qquad (2)$$

Since $\lim_{\alpha\to 1}^{\alpha} V(Q) = V(Q)$, Tsalli's [37] entropy given by (2) is called one-parameter generalization of Shannon's [3] entropy (1948).

Bookee and Vander Lubbe [38] (1980) studied R-norm information measure of a discrete probality distribution as :

$$V_R(Q) = \frac{R}{R-1}\left[\left(1-\sum_{i=1}^{m} q_i^{R}\right)^{\frac{1}{R}}\right] \qquad (3)$$

where $0 < R < \infty$. At $R = 1$, $V_R(Q)$ does not exist; accordingly, we use L'Hospital rule to prove that $V_R(Q)$ access the Shannon's [3] entropy as $R \to 1$; i.e, $\lim_{R\to 1} V_R(Q) = V(Q) - \sum_{i=1}^{m}(q_i)\log(q_i)$ which is Shannon's [3] entropy (1948). Moreover, R-norm entropy was parametrically generalized to (R, S)-norm entropy by Joshi and Kumar [?] (2016 a, b). In particular, there is also a close relationship between the Renyi's entropy and R-norm entropy written as

$$V^R(Q) = \frac{1}{1-R}\log\left(1+\frac{1-R}{R}Q_R(Q)\right) = \frac{1}{1-R}\log_D \sum_{i=1}^{m} q_i^{R} \qquad (4)$$

where $V^R(Q)$ is the Renyi's entropy.

Although, a primary contrast exists, the Shannon and Renyi's entropy are additive whereas the R-norm entropy is pseudo additive.

Remark: 1. If $\alpha = 2$, then the measure (2) becomes Ginni-Simpson's index of diversity:

$$^{2}V(Q) = \left[1-\sum_{i=1}^{m} q_i^{2}\right] \qquad (5)$$

2. If R= 2, (4) becomes Renyi's Index as:

$$V^2(Q) = \log_D\left(\sum_{i=1}^{m}\frac{1}{q_i^2}\right) \qquad (6)$$

Now, we present some definitions related to the FS theory. Here, $X = \{x_1, x_2, \ldots, x_m\}$ will denote the universe of discourse and FS(X) will denote the set of all FSs of the universe of discourse X.

Definition 1. (Zadeh [25]). An FS A in X is given by $A = \{\langle x_i, \mu_A(x_i)\rangle / x_i \in X\}$, where $\mu_A : A \to [0,1]$ is a membership function and $\mu_A(x_i)$ is the membership value of $x_i \in X$ in A.

Definition 2. (Zadeh [25]). Let A, B $\in$ FS(X) be given by $A = \{\langle x_i, \mu_A(x_i)\rangle / x_i \in X\}$ and $B = \{\langle x_i, \mu_B(x_i)\rangle / x_i \in X\}$. Then the following are some FS operations.

1. Union: $A \cup B = \{\langle x_i, \max(\mu_A(x_i), \mu_B(x_i))\rangle / x_i \in X\}$;

2. Intersection: $A \cap B = \{\langle x_i, \min(\mu_A(x_i), \mu_B(x_i))\rangle / x_i \in X\}$;

3. Complement: $\overline{A} = \{\langle x_i, 1-\mu_A(x_i)\rangle / x_i \in X\}$.

Definition 3. (Zadeh [25]). Let A be an FS, then an FS A^* is called a sharpened version of A if the following conditions are satisfied:

$\mu_{A^*} \le \mu_A(x_i)$, if $\mu_A(x_i) \le \frac{1}{2} \forall_i$

$\mu_{A^*} \ge \mu_A(x_i)$, if $\mu_A(x_i) \ge \frac{1}{2} \forall_i$

The FEM is used in fuzzy theory to determine the content of imprecision associated with an FS. De Luca and Termini[26] had formalized the concept of an FEM.

Definition 4. (Luca and Termini [26]). Let A $\in$ FS(X), then $e(A)$, where $e : FS(X) \to R^+ \cup \{0\}$, is called an entropy of FS A if $e(A)$ has the following properties:

(E1) (Maximality): $e(A)$ is maximum if and only if A is most FS, that is, $\mu_A(x_i) = \frac{1}{2} \forall_i$

(E2) (Sharpness): e(A) is minimum if and only if A is a crisp set, that $\mu_A(x_i) = 0$ or $1 \forall_i$

(E3) (Resolution): If A^* is a sharpened version of A, then $e(A) \ge e(A^*)$.

(E4) (Symmetry): If Ā is the complement of A, then e(A) = e(Ā).

Definition 5. (Montes et al. [32]). A function D : FS(X) × FS(X) $\to$ R is called a fuzzy divergence measure if $\forall A_1, A_2, A_3 \in$ FS(X), we have

(D1) $D(A_1 , A_2) = D(A_2, A_1)$;

(D2) $D(A_1, A_1) = 0$;

(D3) $\max (D(A_1 \cup A_3, A_2 \cup A_3), D(A_1 \cap A_3 \cap A_2 \cap A_3) \le D(A_1, A_2)$.

Definition 6. (Xuecheng [13]). A function S : FS(X) × FS(X) $\to R^+$ is called a fuzzy similarity measure if $\forall A_1, A_2, A_3 \in$ FS(X), and B $\in$FS(X) (power set of X), we have

(S1) $S (A_1, A_2) = S(A_2, A_1)$;

(S2) S(B, B) = 0;

(S3) $S(A_1, A_2) = \max_{A_2, A_3} S(A_2, A_3)$;

(S4) If $A_1 \subseteq A_2 \subseteq A_3$, then $S(A_1, A_2) \ge S(A_1, A_3)$ and $S(A_2, A_3) \ge S(A_1, A_3)$.

3 Sine Information Measure

Definition 7. Let _m be a complete probability distribution, then for some $Q \in \Delta_m$, the Sine entropy is defined by

$$V_S(Q) = \sum_{i=1}^{m} \frac{\sin(\pi q_i)}{2}, 0 \le q_i \le 1 \tag{7}$$

Note that sin(πq) is a symmetric function with respect to q = 1/2, and it is strictly increasing in [0, 1/2] and strictly decreasing in [1/2, 1]. The sine entropy $V_S(Q)$ proposes a good measure to characterize the uncertainty of an event. For an event with q = 0, which means Q happens almost impossibly, we have $V_S(Q)$ = 0. That is, the event possesses no uncertainty. For an event with q = 1, which means event happens almost surely, we have $V_S(Q)$ = 0. That is, the event possesses no uncertainty, either. For an event with q = 1=2, in which case it is the most difficult to predict Q, we have *VS(Q)* = 1. That is, the event possesses the most uncertainty. In the next section, we will purpose a new generalized fuzzy entropy measure corresponding to 7 known as Sine fuzzy information measure and verify the axiomatic requirements.

3.1 A Generalized Fuzzy Entropy Measure

In this section, we purpose a new generalized fuzzy entropy measure called fuzzy sine information measure and discuss its advantages.

Let A $\in$ FS(X), we define fuzzy sine information measure as

$$H_s(A) = \frac{1}{2}\sum_{i=1}^{m} \sin(\pi\mu(x_i)) + \sin(\pi(1-\mu(x_i))), 0 \le \mu(x_i) \le 1. \quad (8)$$

Theorem 1. The measure given in Equation 9 is a valid measure.

Proof. To prove the validity of $H_s(A)$, we examine the axiomatic requirements of the measure given in the definition 4.

(E1) Let $H_s(A) = \sum_{i=1}^{m} h(\mu(x_i)), 0 \le \mu(x_i) \le 1.$

where

$$h(\pi\mu(x_i)) = \sin(\pi\mu(x_i)) + \sin(\pi(1-\mu(x_i))),\ 0 \le \mu(x_i) \le 1. \quad (9)$$

Differentiating Equation with respect to $\mu(x_i)$, we get

$$\frac{\partial h(\mu(x_i)}{\partial \mu(x_i)} = \pi\cos(\pi\mu(x_i)) - \pi\cos(\pi(1-\mu(x_i))).$$

Let 0 $\le \mu(x_i) < .5$, then

$$\frac{\partial h(\mu(x_i)}{\partial \mu(x_i)} > 0,$$

Let 0.5 $< \mu(x_i) \le 1$, then

$$\frac{\partial h(\mu(x_i)}{\partial \mu(x_i)} < 0,$$

and for $\mu(x_i) = 0.5$,

$$\frac{\partial h(\mu(x_i)}{\partial \mu(x_i)} = 0.$$

This shows that $H_s(A)$ has the maximum value if and only if A is the most FS.

(E2) Let A be a crisp set, then

$$\mu(x_i) = 0 \text{ or } 1 \forall x_i \in A$$

For $\mu(x_i) = 0$,

$$H_s(A) = \sum_{i=1}^{m} \sin(0) + \sin(\pi(1-0)) = 0.$$

For $\mu(x_i) = 1$,

$$H_s(A) = \sum_{i=1}^{m} \sin(\pi) + \sin(\pi(1-1)) = 0.$$

Thus, $H_s(A) = 0$ when A is a crisp set.

Conversely, assume that $H_s(A) = 0$

$$\Rightarrow \sum_{i=1}^{m} \sin(\pi\mu(x_i)) + \sin(\pi(1-\mu(x_i))) = 0. \quad (10)$$

Since $0 \le x \le 1$, the above equation will hold only if $\mu(x_i) = 0$ or $1 \forall x_i \in A$, that is A is a crisp set.

Hence, $H_s(A)$ is minimum if and only if A is a crisp set.

(E3) Since in [0, 0:5), $H_s(A)$ is a increasing function of $\mu(x_i)$ and is a decreasing function of $\mu(x_i)$ in (0:5; 1], so

$\Rightarrow H_s(A^*) \le H_s(A)$ in $[0, 0.5)$

$\Rightarrow H_s(A^*) \le H_s(A)$ in [0.5,1)

Hence,

$H_s(A^*) \le H_s(A).$

(E4) $H_s(A) = H_s(\overline{A})$ follows from the definition of $H_s(A)$.

Hence, $H_s(A)$ is a valid FEM.

Theorem 2. For A, B $\in$ FS(U),

$$H_s(A\cup B) + H_s(A\cap B) = H_s(A) + H_s(B), \qquad (11)$$

Proof. Let $X = \{x_1, x_2, ..., x_m\}, X_1 = \{x_i \in U / A \subset B\},$ and $X_2 = \{x_i \in U / A \supseteq B\}.$

That is, $\mu_A(x_i) < \mu_B(x_i)$ for all $\mu_i \in X_1$ and $\mu_A(x_i) \ge \mu_B(x_i)$ for all $x_i \in X_2$.

Now,

$$H_s(A\cup B) = \sum_{i=1}^{m} \sin(\pi\mu_{A\cup B}(x_i)) + \sin(\pi(1-\mu_{A\cup B}(x_i))),\ 0 \le \mu(x_i) \le 1.$$

$$= \frac{1}{m}\left(\sum_{X_1} \sin(\pi\mu_B(x_i)) + \sin(\pi(1-\mu_B(x_i))) + \sum_{X_2} \sin(\pi\mu_A(x_i)) + \sin(\pi(1-\mu_A(x_i)))\right) \qquad (12)$$

Similarly,

$$H_s(A\cap B) = \sum_{i=1}^{m} \sin(\pi\mu_{A\cap B}(x_i)) + \sin(\pi(1-\mu_{A\cap B}(x_i))), 0 \le \mu(x_i) \le 1.$$

$$= \frac{1}{m}\left(\sum_{X_1} \sin(\pi\mu_A(x_i)) + \sin(\pi(1-\mu_A(x_i))) + \sum_{X_2} \sin(\pi\mu_B(x_i)) + \sin(\pi(1-\mu_B(x_i)))\right) \qquad (13)$$

Adding above equations, we get

$$H_s(A\cup B) + H_s(A\cap B) = H_s(A) + H_s(B)$$

3.2 Comparison with other measures

Now, we compare the proposed Sine-fuzzy entropy measure with the other measures that are used in literature, we can find the benefits of the proposed measure by exploring this comparison. Here, we manipulate the values of structure linguistic variables, compute the ambiguity content of fuzzy sets, and compute the attribute weights for MCDM issues.

These are some entropies in fuzzy environment that are listed below:

Verma R and Sharma BD [9] defined fuzzy entropy for a fuzzy set A as

$$E_{VS} = \frac{1}{m(e^{1-0.5^\alpha} - 1}\sum_{i=1}^{m}\left(\mu_A(\upsilon_i)e^{(1-\mu_A(\upsilon_i))^\alpha)} + (1-\mu_A(\upsilon_i))e^{(1-(1-\mu_A(\upsilon_i))^\alpha)} - 1\right), \alpha > 0 \qquad (14)$$

Joshi R and Kumar s [?] introduced exponential fuzzy entropy of order α for fuzzy set A as

$$E_{JK} = \frac{1}{m(e^{1-0.5^\alpha} - 1}\sum_{i=1}^{m}\begin{pmatrix}\mu_A(\upsilon_i)e^{(1-\mu_A(\upsilon_i))^\alpha)} + (1-\mu_A(\upsilon_i))e^{(1-(1-\mu_A(\upsilon_i))^\alpha)} \\ -\mu_A(\upsilon_i)e^{(1-\mu_A(\upsilon_i))^\beta)} + (1-\mu_A(\upsilon_i))e^{(1-(1-\mu_A(\upsilon_i))^\beta))}\end{pmatrix} \qquad (15)$$

Depending on exponential function, Pal and Pal [8] proposed exponential fuzzy entropy for fuzzy set A as

$$E_{PP} = \frac{1}{\sqrt{e}-1}\sum_{i=1}^{m}\left(\mu_A(u_i)e^{(1-\mu_A(u_i))} + (1-\mu_A(u_i))e^{(\mu_A(u_i))} - 1\right) \qquad (16)$$

Further Kapur [11] introduced a fuzzy information measure as

$$E_K = \frac{1}{(1-\alpha)}\sum_{i=1}^{m}\left(((\mu_A(u_i))^\alpha + (1-\mu_A(u_i))^\alpha) - 1\right), \alpha \ne 1, \alpha > 0 \qquad (17)$$

Later Parkash [12] introduced a new parametric measure of fuzzy entropy as

$$E_P = \frac{1}{(1-\alpha)^\beta}\sum_{i=1}^{m}\left(((\mu_A(u_i))^\alpha + (1-\mu_A(u_i))^\alpha)^\beta - 1\right), \alpha \neq 1, \alpha > 0, \beta \neq 0 \quad (18)$$

And our proposed measure is

$$H_s(A) = \frac{1}{2}\sum_{i=1}^{m}\sin(\pi\mu(x_i)) + \sin(\pi(1-\mu(x_i))), 0 \leq x \leq 1. \quad (19)$$

3.2.1 Structured linguistic comparison

The modifiers of linguistic variables are represented with the help of linguistic hedges, like, slightly, more or less, very, and so forth. Usually, FSs are used as linguistic variables and therefore the linguistic hedges may be regarded as FS operations. Here we compare the performance of the proposed GFEM with other information measures of FSs by considering linguistic hedges as FS operations.

Let $A = \{(\mu, \mu_A(\mu)) / \mu \in U\}$ be an FS, then $A^m = \{(\mu, \mu_A(\mu)) / \mu \in U\}$ is the modifier of FS A.

For an FS A, two operations, namely, concentration and dilation, were introduced by Zadeh [25] and are given as CON(A) = A^2 and DIL(A) = $A^{½}$.

The concentration and dilation of an FS A are the mathematical models often used for modifiers. So, the linguistic hedges on an FS A can be defined with the help of these mathematical models as follows:

More or less A = DIL(A) = $A^{½}$, Very A= CON(A) = A^2, Quite Very A = A^3, and Very Very A = A^4.

For clarity, we use the following abbreviations: M/L is for More or Less, V for very, Q. V. for Quite Very and V.V for Very Very. Fuzzy hedges for fuzzy set A is defined by

$$\begin{cases} M/L & \text{represented } A^{1/2} \\ V & \text{represented } A^2 \\ Q.V & \text{represented } A^3 \\ V.V & \text{represented } A^4 \end{cases}$$

We compare various fuzzy information measures by consideration of these FSs. The entropy measures of such FSs should have the following order for good performance.

$$e(A^{½}) > e(A) > e(A^2) > e(A^3) > e(A^4). \quad (20)$$

We establish the superiority of the suggested GFEM with the help of the following illustrative example.

Example 1 (Garg et al. [?]). Let A ={(0.1), (0.3), (0.5), (0.9), (1.0) be an FS in the universe of discourse U = {u_1, u_2, u_3, u_4, u_5}.

Using the definition of the modifier of an FS, we have

$$A^{1/2} = \{(0.3162), (0.5477), (0.7071), (0.9487), (1.0000)\},$$

$$A^2 = \{0.0100), (0.0900), (0.500), (0.8100), (1.0000)\},$$

$$A_3 = \{(0.0010), (0.0270), (0.1250), (0.7290), (1.0000)\},$$

$$A_4 = \{(0.0001), (0.0081), (0.625), (0.6561), (1.0000)\}.$$

Due to the characterization of linguistic variables, we regard A as Large on U, so the hedges represented by the above FSs are described as follows:

$A^{½}$ may be treated as More or Less Large, A^2 may be treated as Very Large, A^3 may be treated as Quite Very Large, A^4 may be treated as Very Very Large.

For comparison, we utilize the existing FEMs given in Table 1 together with our proposed one and the results are listed in Tables 1-6.

Table 1. Entropy measure values by existing methods regarding

Entropy measure	*(M/L)*	(A)	(V)	(Q.V)	(V.V)
E_{jk}					
$\alpha = 0.5$	0.5983	0.5449	0.3835	0.2963	0.2526
$\alpha = 1.0$	0.5789	0.5176	0.3530	0.2722	0.2367
$\alpha = 1.5$	0.5671	0.4996	0.3325	0.2568	0.2276
$\alpha = 2.0$	0.5607	0.4911	0.3225	0.2494	0.2235
$\alpha = 2.5$	0.5594	0.4896	0.3206	0.2481	0.2227

Table 2. Entropy measure values by existing methods regarding Example 1

Entropy measure	(M/L)	(A)	(V)	(Q.V)	(V.V)
^{E}jk					
$\alpha = 1.1, \beta = 0.1$	-0.09995	-0.8857	-0.5962	-0.4594	-0.4028
$\alpha = 1.1, \beta = 0.2$	-0.7439	-0.6562	-0.4381	-0.3373	-0.2972
$\alpha = 2.0, \beta = 0.9$	-0.5952	-0.5067	-0.3135	-0.2436	-0.2286
$\alpha = 0.1, \beta = 1.1$	0 2227	0.1973	0.1328	0.1023	0.0897
$\alpha = 0.5, \beta = 2.0$	0.5594	0.4896	0.3206	0.2481	0.2227
$\alpha = 0.9, \beta = 5.0$	0.3033	0.2709	0.1894	0.1450	0.1236

Table 3. **Entropy measure values by existing methods regarding Example 1**

Entropy measure	(M/L)	(A)	(V)	(Q.V)	(V.V)
E_{pp}	1.0945	0.9771	0.6664	0.5138	0.4468

Table 4. **Entropy measure values by existing methods regarding Example 1**

Entropy measure	(M/L)	(A)	*(V)*	*(Q.V)*	(V.V)
E_K					
$\alpha = 0.5$	2.7689	2.6568	2.1017	1.6904	1.4211
$\alpha = 1.5$	1.7169	1.5441	1.0651	0.8221	0.7102

$\alpha = 5.5$	0.6505	0.5986	0.4308	0.3309	0.2760
$\alpha = 10.5$	0.3356	0.3488	0.2706	0.2082	0.1645
$\alpha = 13.0$	0.2898	0.2901	0.2284	0.1766	0.1388
$\alpha = 15.0$	0.1927	0.1977	0.1586	0.1248	0.0988

Table 5. Entropy measure values by existing methods regarding Example 1

Entropy measure		***(M/L)***	**(A)**	**(V)**	**(Q.V)**	**(V.V)**
E_P						
$\alpha = 0.5, \beta$	$=1.5$	3.0088	2.8745	2.2541	1.8081	1.5219
$\alpha = 2.0, \beta$	$=2.5$	1.0159	0.9397	0.6808	0.5229	0.4339
$\alpha = 3.5, \beta$	$=3.7$	0.3751	0.3756	0.2956	0.2286	0.1796
$\alpha = 4.5, \beta$	$=4.0$	0.2578	0.2641	0.2110	0.1652	0.1303
$\alpha = 5.5, \beta$	$=5.0$	0.1673	0.1729	01406	0.1125	0.0904
$\alpha = 6.5, \beta$	$=5.5$	0.1272	0.1307	0.1080	0.0876	0.0713

Table 6. Our proposed entropy measure values regarding.

Entropy measure		***(M/L)***	**(A)**	**(V)**	**(Q.V)**	**(V.V)**
H_S	$=1.5$	0.5565	0.4854	0.3159	0.2445	0.2205

From above Tables, we observe the following:

1. The FEMs E_{VS}, E_{PP}, satisfy the desired requirements given in Equations (20).
2. The FEM E_{JK} satisfies the desired requirement given in Equation (20) only when $0 < \alpha < 1$, $\beta > 1$ and not when $\alpha > 1$, $0 < \beta < 1$.
3. The FEM E_K satisfies the requirement given in Equation (20) only when $\alpha < 13$ and not when $\alpha \geq 13$.
4. The FEM E_P satisfies the desired requirement given in Equation (20) for $\alpha < 4$ and $\beta < 3.5$ and not when $\alpha \geq 4$ and $\beta \geq 3.5$.
5. Our suggested FEM HS follows the desired condition given in Equation (20).

3.2.2 Calculation of Ambiguity

An FEM quantifies the content of fuzziness in an FS, whereas an FKM quantifies the precision/sharpness in an FS. Two different FSs have different ambiguity content unless they are the complement of each other. However, by some existing FEMs, the ambiguity content of two different FSs comes out to be the same. Such a situation is fuzzy theoretically unreasonable. So, there is always a need to introduce new measures or to generalize the existing ones for addressing such an issue. Here, we consider a numerical example related to the ambiguity content of FSs, that shows the effectiveness of our proposed FEM H_S(A).

Example 2. Consider six differnt FSs $A_i(i = 1, 2, 3, 4, 5, 6)$ in the universe of discourse $U = \{u_1, u_2, u_3, u_4, u_5\}$ given as

A_1 = {0.230, 0.314, 0.564, 0.215, 0.126},

A_2 = {0.904, 0.200, 0.259, 0.483, 0.651},

A_3 = {0.525, 0.521, 0.152, 0.235, 0.142},

A_4 = {0.066, 0.500, 0.525, 0.142, 0.543},

A_5 = {0.582, 0.591, 0.015, 0.658, 0.434},

A_6 = {0.415, 0.215, 0.399, 0.024, 0.573}.

We now calculate the ambiguity content of these FSs with the help of some existing FEMs and also with the help of our proposed FEM. The detailed results are shown in Table 7.

From Table 7, we see that some existing FEMs could not differentiate between

Table 7. Ambiguity computation by various fuzzy compatibility measures regarding Example 2.

	A_1	A_2	A_3	A_4	A_5	A_6
E_{VS}	0.4038	**0.4463**	0.4152	**0.4463**	0.5015	0.4317
E_{JK}	0.0722	**0.0726**	**0.0726**	0.0757	0.0813	0.0755
E_{pp}	1.3997	**1.3955**	1.3982	1.4139	1.4695	**1.3955**
E_K	**1.8346**	1.8312	**1.8346**	1.8620	1.9409	1.8389
E_P	1.2167	1.1998	1.2044	**1.1716**	**1.1716**	1.1444
$H_S(A)$(proposed)	0.6972	0.6999	0.7012	0.7250	0.7662	0.7178

different FSs in terms of ambiguity content. However, our proposed FEM, easily differentiate these different FSs. Thus, it is evident from this example that there is always a need to formulate new FEMs.

3.2.3 Computation of Attribute weight

The attribute weights contribute significantly in the process of selecting a suitable alternative in an MADM problem. Usually, the decision-experts provide the attribute weights, and these attribute weights are termed subjective weights. However, if the attribute weights are not pre-assigned by the decision-experts, then the attribute weights are derived using some models and such attribute weights are termed as objective weights. Here, we show that the entropy-based attribute weight method does not always give appropriate results and in such situations, our proposed entropy measure could be utilized for determining the attribute weights more effectively. We consider a numerical example related to the attribute weight computation in an MADM problem.

Example 3. Consider a fuzzy decision matrix D_1 on five alternatives $A_i(i = 1, 2, 3, 4, 5, 6)$, and six attributes $C_j(j = 1, 2, 3, 4, 5, 6)$ given as

$$D_{1}= \begin{pmatrix} 0.152 & 0.015 & 0.564 & 0.399 & 0.259 & 0.525 \\ 0.142 & 0.434 & 0.126 & 0.573 & 0.651 & 0.543 \\ 0.525 & 0.582 & 0.230 & 0.415 & 0.904 & 0.066 \\ 0.213 & 0.658 & 0.215 & 0.024 & 0.483 & 0.142 \\ 0.521 & 0.591 & 0.314 & 0.215 & 0.200 & 0.500 \end{pmatrix}$$

Now, we calculate the attribute weights with the help of the following method.

1. Entropy-based attribute weight determining method:

$w_j = \frac{1-E(C_j)}{\sum_{j=1}^{6} 1-E(C_j)}$, j = 1, 2, 3, 4, 5, 6, where E demotes FEM.

Table 8. Attribute weights regarding Example 3.

	w_1	w_2	w_3	w_4	w_5	w_6
E_{VS}	0.1743	**0.1486**	0.1777	0.1694	**0.1650**	**0.1650**
E_{JK}	**0.1644**	0.1729	0.1640	0.1669	**0.1644**	0.1674
E_{PP}	0.1611	0.1899	0.1617	**0.1600**	**0.1600**	0.1674
E_K	**0.1623**	0.1830	**0.1623**	0.1631	0.1616	0.1676
E_P	0.1844	**0.1548**	0.1955	0.1302	0.1802	**0.1548**
H_S	0.1765	0.1380	0.1789	0.1667	0.1772	0.1624

With the help of these two-attribute weight determining methods, the weights of attributes are listed in Table 8. From Table 8, we observe that in some situations the attribute weights calculated with the help of some existing FEMs are not reasonable as two different attributes are assigned the same weights and thus affecting the selection of the best alternative. So, we see that our proposed FEM H_S is capable to assign different weights to all the attributes. From this, it can be concluded that the generalization of existing FEMs is always desirable and also the presence of parameters in an FEM provides more freedom to a decision-maker in selecting the best alternative in an MADM problem.

4 A New Jensen-Sine Divergence Measure

Definition 8. For some P, Q $\in \Delta_m$, we defne a new divergence measure as.

$$V\hat{G}S(P,Q) = \sum_{i=1}^{m} \frac{1}{2}(\sin(p_i)\pi + (\sin(q_i)\pi) - \sin\left(\frac{p_i + q_i}{2}\right)\pi \tag{21}$$

To justify the existence, we prove that proposed measure satisfies the following properties.

Properties of $V\hat{G}S(P, Q)$. Some major properties of proposed divergence measure are

1. $V\hat{G}S(P, Q) \geq 0$ with equality when C = D;
2. $V\hat{G}S(P, Q)$ is a convex function of C and D.

Proof. First we prove that $V\hat{G}S(P, Q) \geq 0$.

when $p_i = q_i$

$$V\hat{G}S(P,Q) = \sum_{i=1}^{m} \frac{1}{2}(\sin(p_i)\pi + (\sin(q_i)\pi) - \sin\left(\frac{2p_i}{2}\right)\pi = \sin(p_i)\pi - \sin(p_i)\pi$$

$V\hat{G}S(P, Q) = 0$

Now, we prove the convexity of (21),
For this, we consider a function given by

$$f(x,y)=\sin\pi(x)+\sin\pi(y)-\sin\left(\frac{x+y}{2}\right)\pi \quad (22)$$

Definition 3. The Hessian matrix of a function f of two variables x and y is defined as

$$Hessian(f)=\begin{bmatrix}\frac{\delta^2 f}{\delta x^2} & \frac{\delta^2 f}{\delta x\delta y}\\ \frac{\delta^2 f}{\delta y\delta x} & \frac{\delta^2 f}{\delta y^2}\end{bmatrix} \quad (23)$$

The function *f* is said to be convex at any point in its domain if *Hessian*(*f*) is positive semi definite and concave if *Hessian*(*f*) is negative semi definite at that point. Now differentiating (22) partially with respect *x* and *y*, we get

$$\frac{\delta f}{\delta x}=\pi\cos\pi(x)-\frac{\pi}{2}\cos\left(\frac{x+y}{2}\right)\pi \quad (24)$$

$$\frac{\delta f}{\delta y}=\pi\cos\pi(y)-\frac{\pi}{2}\cos\left(\frac{x+y}{2}\right)\pi \quad (25)$$

To determine stationary point, we substitute $\delta f0/\delta x = 0$. This gives $x = y$ as a stationary point. Now, computing the Hessian matrix of f at $x = y$, we have

$$Hessian(f)=\frac{\pi^2}{4}\sin\pi(x)\begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix} \quad (26)$$

which is positive semi definite. This confirms the convex character of *f*.
Extending the idea from probabilistic setting to fuzzy settings, in the next subsection, we introduce a new fuzzy divergence measure based on measure (21).

4.1 Definition

Definition 9. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a fnite universe of discourse. For any $A, B \in FS(X)$, we defne a fuzzy divergence measure $V\hat{G}S(A, B)$ as

$$V\hat{G}S(A,B)=\sum_{i=1}^{m}\frac{1}{2}\frac{\sin(a_i)\pi+\sin(b_i)\pi}{2}-\sin\left(\frac{a_i+b_i}{2}\right)\pi \quad (27)$$

$$+\frac{\sin(1-a_i)\pi+\sin(1-b_i)\pi}{2}-\sin\left(1-\left(\frac{a_i+b_i}{2}\right)\right)\pi$$

4.2 Justification

Now, a very common query that comes in mind is about the validity of the proposed measure. We answer this question by proving that the proposed measure satisfies
(1) non-negativity and (2) convexity.
Proof. We start with proving the non-negativity of (27).
at $a_i = bi$

$$V\hat{G}S(A,A)=\sum_{i=1}^{m}\frac{\sin(a_i)\pi+\sin(a_i)\pi}{2}-\sin\left(\frac{2a_i}{2}\right)\pi+\frac{\sin(1-a_i)\pi+\sin(1-a_i)\pi}{2}$$

$$-\sin\left(1-\left(\frac{2a_i}{2}\right)\right)\pi$$

$$\Rightarrow V\hat{G}S(A,B)=0$$

Since $\mu_A(x_i)+(1-\mu_A(x_i))=1$ and $\mu_B(x_i)+(1-\mu_B(x_i))=1$ for A, $B\in$ $FS(X)$ and $x_i \in X$, therefore, convexity of (31) follows directly from the convexity of (5). This, justifies the existence of (31).

In the next Section, we study some major properties of $V\hat{G}S(P, Q)$.

4.3 Properties Of Proposed Divergence Measure

Measure (31) possesses the following properties.

Theorem. For $A, B, C \in FS(X)$,

1. $V\hat{G}S(A,B)=V\hat{G}S(B,A)$;
2. $V\hat{G}S(A,B)=0$ if and only if $A=B$;
3. $V\hat{G}S(A,A^c)=0$ if and only $\mu_{A^c}(x_i)=1-\mu_A(x_i)$ for all $x_i \in X$;
4. $V\hat{G}S(A,A\cap B)=V\hat{G}S(A\cap B,B)\le V\hat{G}S(A,B)$
5. $V\hat{G}S(A\cup B,A\cap B)=V\hat{G}S(A\cap B,A\cup B)=V\hat{G}S(A,B)$
6. $V\hat{G}S(A\cap B,A\cup B)=V\hat{G}S(A,B)$
7. $V\hat{G}S(A,A\cup B)+V\hat{G}S(A,A\cap B)=V\hat{G}S(A,B)$
8. $V\hat{G}S(B,A\cup B)+V\hat{G}S(B,A\cap B)=V\hat{G}S(B,B)$
9. $V\hat{G}S(A\cup B,C)\le V\hat{G}S(A,C)+V\hat{G}S(B,C)$
10. $V\hat{G}S(A\cap B,C)\le V\hat{G}S(A,C)+V\hat{G}S(B,C)$
11. $V\hat{G}S(A\cup B,C)\le V\hat{G}S(A\cap B,C)=V\hat{G}S(A,C)+V\hat{G}S(B,C)$
12. $V\hat{G}S(A,B)=V\hat{G}S(A^c,B^c)$
13. $V\hat{G}S(A,B^c)=V\hat{G}S(A^c,B)$
14. $V\hat{G}S(A,B)=V\hat{G}S(A^c,B)=V\hat{G}S(A^c,B^c)+V\hat{G}S(A,B^c)$

where A^c and B^c are the compliments of FSs A and B respectively.

Proof: To prove the properties, we bifurcate X into two parts X_1 and X_2 such that

$X_1 = \{x_i \in X \mid A \subseteq B\}$,

$X_2 = \{x_i \in X \mid A \supseteq B\}$,

(1) Proof of Properties (1), (2), and (3) follows directly from definition (9).

(2) Proof of Property (4)

$$V\hat{G}S(A,A\cup B)=\sum_{i=1}^{m}\frac{\sin(a_i)\pi+\sin(a_i\cup b_i)\pi}{2}-\sin\left(\frac{a_i+a_i\cup b_i}{2}\right)\pi \qquad (28)$$
$$+\frac{\sin(1-a_i)\pi+\sin(1-a_i\cup b_i)\pi}{2}-\sin\left(1-\left(\frac{a_i+(a_i\cup b_i)}{2}\right)\right)\pi$$

This implies,

$$V\hat{G}S(A,A)=\sum_{X_1}\frac{\sin(a_i)\pi+\sin(a_i)\pi}{2}-\frac{\sin(a_i+b_i)}{2}+\frac{\sin(1-a_i)\pi+\sin(1-b_i)\pi}{2}$$
$$-\sin\left(1-\left(\frac{a_i+b_i}{2}\right)\right)\pi+\sum_{X_2}\frac{\sin(a_i)\pi+\sin(a_i)\pi}{2}-\frac{\sin(a_i+a_i)}{2}$$
$$+\frac{\sin(1-a_i)\pi+\sin(1-a_i)\pi}{2}-\sin\left(1-\left(\frac{a_i+a_i}{2}\right)\right)\pi$$

$$\Rightarrow V\hat{G}S(A,B) \le \sum_{X_1} \frac{\sin(a_i)\pi + \sin(b_i)\pi}{2} - \frac{\sin(a_i + b_i)}{2} + \frac{\sin(1-a_i)\pi + \sin(1-b_i)\pi}{2} - \sin\left(1-\left(\frac{a_i + b_i}{2}\right)\right)\pi$$

$$\Rightarrow V\hat{G}S(A,B) \le V\hat{G}S(A,B)$$

Similarly, we can prove that $V\hat{G}S(A \cap B, B) \le V\hat{G}S(A,B)$

(3) Proof of Property (5)

$$V\hat{G}S(A \cup B, A \cap B) = \sum_{X_1} V\hat{G}S(B,A) + \sum_{X_2} V\hat{G}S(A,B) = V\hat{G}S(A,B) \qquad (29)$$

Similarly, we may prove that

$$V\hat{G}S(A \cap B, A \cup B) = \sum_{X_1} V\hat{G}S(B,A) + \sum_{X_2} V\hat{G}S(A,B) = V\hat{G}S(A,B) \qquad (30)$$

(4) Proof of Property (6). Proof follows from Property (5) and Property (1).

(5) Proof of Property (7).

$$V\hat{G}S(A, A \cup B) + V\hat{G}S(A, A \cap B) = \sum_{X_1} V\hat{G}S(A,B) + \sum_{X_2} V\hat{G}S(A,A) + \sum_{X_1} V\hat{G}S(A,A) + \sum_{X_2} V\hat{G}S(A,B);$$

$$= V\hat{G}S(A,B) \qquad (31)$$

(6) Proof of Property (8). Proof of Property (8) is similar to that of Property (7).

(7) Proof of Property (9).

$$V\hat{G}S(A \cup B, C) = \sum_{i=1}^{m} \frac{\sin(a_i \cup b_i)\pi + \sin(c_i)\pi}{2} - \sin\left(\frac{a_i \cup b_i + c_i}{2}\right)\pi + \frac{\sin(1-(a_i \cup b_i))\pi + \sin(1-c_i)\pi}{2} - \sin\left(1-\left(\frac{(a_i \cup b_i) + c_i}{2}\right)\right)\pi \qquad (32)$$

$$\Rightarrow V\hat{G}S(A \cup B, C) = \sum_{X_1} \frac{\sin(b_i)\pi + \sin(c_i)\pi}{2} - \sin\left(\frac{b_i + c_i}{2}\right)\pi + \frac{\sin(1-b_i)\pi + \sin(1-c_i)\pi}{2} - \frac{\sin(1-(b_i + c_i))\pi}{2} + \sum_{X_2} \frac{\sin(a_i)\pi + \sin(c_i)\pi}{2} - \sin\left(\frac{a_i + c_i}{2}\right)\pi + \frac{\sin(1-a_i)\pi + \sin(1-c_i)\pi}{2} - \sin\left(1-\left(\frac{a_i + c_i}{2}\right)\right)\pi$$

$$\Rightarrow V\hat{G}S(A \cup B, R) \le V\hat{G}S(B,C) + V\hat{G}S(A,C)$$

(8) Proof for Property (10). Similarly, we can prove Property (10).

(9) Proof of Property (11).

$$V\hat{G}S(A \cup B, C) + V\hat{G}S(A \cap B, C) = \sum_{X_1} V\hat{G}S(B,C) + \sum_{X_2} V\hat{G}S(A,C) + \sum_{X_1} V\hat{G}S(A,C) + \sum_{X_2} V\hat{G}S(B,C);$$

$$= V\hat{G}S(A,C) + V\hat{G}S(B,C) \qquad (33)$$

(10) Proofs of Properties (12), (13), and (14) follows directly from defnition.

4.4 Application of Proposed Fuzzy Divergence Measure in Pattern Recognition

In pattern recognition problems, a common issue is the presence of vague and uncertain information. Here, we contrast the proposed divergence measure with the existing compatibility measures in pattern recognition. Broadly, a problem related to pattern classification can be formulated in the following manner.

Scenario: Let K_1, K_2,…, K_m be given patterns in a database and B be an unknown pattern.

The realization of these patterns is given in the form of FSs concerning a universal set $X = \{x_1, x_2, ..., x_m\}$. We have the following realizations:

$$K_i = \left\{< x_i, \mu_{K_i}(x_i) > / x_i \in X)\right\}, i = 1, 2, ...m$$

$$L = \left\{< x_i, \mu_L(x_i) > / x_i \in X)\right\}.$$

Problem: To classify the unknown pattern L to one of the known patterns $K_i(i = 1, 2,.., m)$.

Recognition or classification principle: The pattern L can be classified to a pattern $K_i(i = 1, 2,…, m)$ available in the database in view of certain compatibility measures as follows:

1. Similarity/correlation/accuracy method:

Let $S(K_i, L)/C(K_i, L)/ I(K_i, L), i = 1, 2, ..., m)$ denote the similarity/correlation/accuracy between K_i ($i = 1, 2,…, m$) and L, then L is assigned to K_{i^*}, where $i^* = \text{argmax } S(K_i, L)/ C(K_i, , L)/ I(K_i, L), i = 1, 2, ..., m$

2. Dissimilarity/divergence/distance method:

Let $DS(K_i, L)/D(K_i, L)/DI(K_i, L), i = 1, 2,…, m$ denote the dissimilarity/divergence/distance between $K_i(i =1, 2,…, m)$ and B, then B is assigned to K_{i^*}, where $i^* = \text{argmin } DS(K_i, L)/D(K_i, L)/DI(K_i,L), i = 1, 2, ..., m$.

Now, we examine various patterns with the help of the suggested divergence measure and contrast the results with some available measures of correlation/divergence/similarity in fuzzy settings. We first list some available fuzzy correlation, fuzzy divergence, and fuzzy similarity measures introduced by various researchers. Divergence measure due to Bhandari and Pal [31]:

$$D_{BP} = (A, B) = \sum_{i=1}^{m} \left((\mu_A(\mu_1) - (\mu_B(\mu_1)) \log \frac{\mu_A(\mu_1)}{\mu_B(\mu_1)} + (\mu_B(\mu_1) - \mu_A(\mu_1)) \log \frac{1-\mu_A(\mu_1)}{1-\mu_B(\mu_1)} \right).$$

Similarity measures due to Pappis and Karacapilidis[16]:

$$S_{PK_1} = \frac{\sum_{i=1}^{m} \min(\mu_A(\mu_i), \mu_B(\mu_i) |)}{\sum_{i=1}^{m} \max(\mu_A(\mu_i), \mu_B(\mu_i) |)},$$

$$S_{PK_2} = 1 - \max(| \mu_A(\mu_i) - \mu_B(\mu_i) |), S_{PK_3} = 1 - \frac{\sum_{i=1}^{m} (| \mu_A(\mu_i) - \mu_B(\mu_i) |)}{\sum_{i=1}^{m} (| \mu_A(\mu_i) + \mu_B(\mu_i) |)}.$$

Similarity measures due to Chen et al.[21]:

$$S_{CYH_1} = 1 - \frac{\sum_{i=1}^{m} (| \mu_A(\mu_i) - \mu_B(\mu_i) |)}{n}, S_{CYH_2} = 1 - \max(\min(\mu_A(\mu_i) - \mu_B(\mu_i))),$$

$$S_{CYH_3} = \frac{\sum_{i=1}^{m} (\mu_A(\mu_i) \times \mu_B(\mu_i)}{\max\left(\sum_{i=1}^{m} (\mu_A(\mu_i) \times \mu_B(\mu_i), \sum_{i=1}^{m} (\mu_A(\mu_i) \times \mu_B(\mu_i))\right)}$$

Similarity measures due to Wang[22]:

$$S_{W_1}(A, B) = \frac{1}{m} \sum_{i=1}^{m} \left(\frac{\min(\mu_A(\mu_i), \mu_B(\mu_i)}{\max(\mu_A(\mu_i), \mu_B(\mu_i)} \right),$$

$$S_{W2}(A,B)=\frac{1}{n}\sum_{i=1}^{m}\left(1-|(\mu_A(\mu_i)-\mu_B(\mu_i)|\right).$$

Correlation measure due to Chiang and Lin[23]:

$$C_{CL}(A,B)=\frac{\dfrac{\sum_{i=1}^{m}(\mu_A(\mu_i)-\bar{\mu_A})(\mu_B(\mu_i)-\bar{\mu_B})}{m-1}}{\left(\dfrac{\sum_{i=1}^{m}(\mu_A(\mu_i)-\bar{\mu_A})^2}{(m-1)}\times\dfrac{\sum_{i=1}^{m}(\mu_A(\mu_i)-\bar{\mu_B})^2}{(m-1)}\right)^{\frac{1}{2}}},$$

$$\bar{\mu_A}=\frac{\sum_{i=1}^{m}\mu_A(\mu_i)}{m}, \text{ and } \bar{\mu_B}=\frac{\sum_{i=1}^{m}\mu_B(\mu_i)}{m}.$$

Correlation measure due to Chaudhuri and Bhattacharya[14]:

$$C_{CB}=1-\sum_{i=1}^{m}\left|\frac{\mu_A(\mu_i)}{\sum_{i=1}^{m}\mu_A(\mu_i)}-\frac{\mu_B(\mu_i)}{\sum_{i=1}^{m}\mu_B(\mu_i)}\right|.$$

The proposed Jensen-Sine fuzzy divergence measure

$$V\hat{G}S(A,B)=\sum_{i=1}^{m}\frac{\sin(a_i)\pi+\sin(b_i)\pi}{2}-\sin\left(\frac{a_i+b_i}{2}\right)\pi$$
$$+\frac{\sin(1-a_i)\pi+\sin(1-b_i)\pi}{2}-\sin\left(1-\left(\frac{a_i+b_i}{2}\right)\right)\pi$$

Now, we solve two numerical examples related to pattern recognition in the fuzzy environment.

Example 4 Consider five known patterns K_i, (i = 1, 2, 3, 4, 5) and an unknown pattern L in the domain $X = \{x_1, x_2, x_3, x_4, x_5\}$ expressed in the form of FSs as

$K_1 = \{(x_1, 0.10), (x_2, 0.80), (x_3, 0.60), (x_4, 0.30), (x_5, 0.20)\}$;

$K_2 = \{(x_1, 0.70), (x_2, 0.50), (x_3, 0.20), (x_4, 0.50), (x_5, 0.10)\}$;

$K_3 = \{(x_1, 0.40), (x_2, 0.60), (x_3, 0.10), (x_4, 0.50), (x_5, 0.20)\}$;

$K_4 = \{(x_1, 0.20), (x_2, 0.30), (x_3, 0.10), (x_4, 0.40), (x_5, 0.50)\}$;

$K_5 = \{(x_1, 0.50), (x_2, 0.10), (x_3, 0.40), (x_4, 0.30), (x_5, 0.50)\}$;

$L = \{(x_1, 0.60), (x_2, 0.30), (x_3, 0.20), (x_4, 0.40), (x_5, 0.50)\}$.

The problem is to classify the unknown pattern B into one of the known patterns K_i(i = 1, 2, 3, 4, 5). For this, we use the above-mentioned fuzzy compatibility measures together with our proposed measure, and the results are listed in Table 9.

It is clear from Table 9 that except S_{PK2}, all other fuzzy compatibility measures including our proposed Jensen-Sine fuzzy divergence measure classify the unknown

Table 9. Values of the existing fuzzy compatibility measures including our suggested divergence measure regarding Example 4

	(K_1,L)	**(K_2 ,L)**	**(K_3, L)**	**(K_4, L)**	**(K_5, L)**	**Classification result**
S_{PK1}	0.4074	0.7273	0.6364	0.6500	0.6364	K_1
S_{PK2}	0.5000	**0.8000**	0.7000	0.6000	**0.8000**	Not classified
S_{PK3}	0.5789	0.8421	0.7778	0.7879	0.7778	K_1
S_{CHY1}	0.6800	0.8800	0.8400	0.8600	0.8400	K_1
S_{CHY2}	0.3000	0.6000	0.4000	0.4000	0.5000	K_1

S_{CHY3}	0.5263	0.8077	0.8537	0.7297	0.8947	K_1
S_{W1}	0.4583	0.7181	0.6267	0.6867	0.6033	K_1
S_{W2}	0.6800	0.8800	0.8400	0.8600	0.8400	K_1
C_{CL}	0.1556	0.7444	0.5556	0.5111	0.5556	K_1
C_{CB}	-0.6785	0.8076	0.4134	0.0000	0.3152	K_1
I_{SG}	2.2507	2.3461	2.3158	2.2679	2.3118	K_1
$V\hat{G}S$ *(A,B) (proposed)*	2.9318	3.9749	3.7774	3.7906	4.2989	K_1

pattern B to the known pattern K_1. This shows that the results due to our suggested Jensen-Sine fuzzy divergence measure are consistent with the existing fuzzy compatibility measures. Next, we consider another example related to pattern recognition in a fuzzy environment.

Example 5. Consider five known patterns K_i; (i = 1, 2, 3,4, 5) and an unknown pattern L in the domain $X = \{x_1, x_2, x_3, x_4, x_5\}$ expressed in the form of FSs as

K_1 = {(xi, 0.500), (x_2, 0.200), (x_3, 0.600), (x_4, 0.100), (x_5, 0.300)},

K_2 = {(x_1, 0.200), (x_2, 0.100), (x_3, 0.700), (x_4, 0.500), (x_5, 0.500)},

K_3 = {(x_1, 0.100), (x_2, 0.220), (x_3, 0.400), (x_4, 0.400), (x_5, 0.600)},

K_4 = {(x_1, 0.110), (x_2, 0.300), (x_3, 0.202), (x_4, 0.400), (x_5, 0.304)},

K_5 = {(x_1, 0.200), (x_2, 0.500), (x_3, 0.500), (x_4, 0.500), (x_5, 0.100)},

L = {(x_1, 0.200), (x_2, 0.300), (x_3, 0.600), (x_4, 0.400), (x_5, 0.300)}.

The problem is to classify the unknown pattern L into one of the known patterns K_i(i = 1, 2, 3,4, 5). For this, we use the above-mentioned fuzzy compatibility measures together with our proposed measure, and the results are listed in Table 10.

Table 10. Values of the existing fuzzy compatibility measures including our suggested divergence measure regarding Example 5

	(K_1,L)	**(K_2 ,L)**	**(K_3, L)**	**(K_4, L)**	**(K_5, L)**	**Classification result**
S_{PK1}	0.7143	0.6762	0.7273	0.7273	0.6667	K_5
S_{PK2}	0.8000	0.7000	0.8000	0.6020	0.7000	K_4
S_{PK3}	0.8333	0.8068	0.8421	0.8421	0.8000	K_5
S_{CHY1}	0.8800	0.8640	0.8800	0.9016	0.8600	K_5
S_{CHY2}	0.5000	**0.4000**	0.6000	**0.4000**	0.6000	Not classified
S_{CHY3}	0.9000	0.9000	0.8077	0.6546	0.8667	K_4
S_{W1}	0.7133	0.6800	0.7181	0.7747	0.6633	K_5
S_{W2}	0.8800	0.8640	0.8800	0.9016	0.8600	K_5
C_{CL}	0.6667	0.6150	0.7444	0.5853	0.5752	K_5
C_{CB}	0.6089	0.4028	0.8076	0.1586	0.3021	K_4
I_{SG}	1.6870	1.6757	1.7107	1.6610	1.6747	K_4
$V\hat{G}S$ *(A,B)*	3.6995	3.8084	3.7858	3.6232	4.1079	K_4

*(pro-
posed)*

It is clear from Table 10 that except S_{CHY2}, all other fuzzy compatibility measures including our proposed Jensen-Sine fuzzy divergence measure classify the unknown pattern B to the known pattern K_i, (i = 4, 5).

Thus, from Examples 4 and 5, we conclude that the performance of our proposed Jensen-Sine fuzzy divergence measure is consistent and better than some of the existing fuzzy compatibility measures.

5. Application Of Proposed Fuzzy Divergence Measure In Decision-Making

In recent years, fuzzy divergence measures have been extensively applied by many researchers in many felds like in image thresholding, bioinformatics, etc. [5, 811]. In this paper, we apply the proposed fuzzy divergence measure in strategic decision-making. In strategic decision-making problems, we have a set of strategies say Λ_1, Λ_2, …, Λ_n and another set of inputs say Ω_1, Ω_2, …, Ω_m Our aim is to fnd a particular strategy which if applied with a specifc input produces optimum results. Degree of efectiveness of a particular strategy corresponding to specifc input can be yielded by the method suggested by Liu and Wang [23] as follows.

Consider a team comprised of N-number of experts.Team members are expected to respond in the form yes if he/she supports a particular strategy coupled with a specifc input and noin case he/she does not feel the particular strategy to be applied with a specifc input. Let the number of experts who support the strategy Λj; (j = 1, 2,…, n) to be applied with input Ω_i; (i = 1, 2, …, m) be denoted by *"$n_{yes}(i, j)$"*. Then the effectiveness of strategy Λ_j with input Ω_i can be calculated as

$$\mu_{\Omega_i}(\Lambda_j) = \frac{n_{yes}(i, j)}{N} \tag{34}$$

Now, let the degree of effectiveness of $\Lambda_j's; (j = 1, 2, ..., n)$ at uniform inputs be denoted by fuzzy set P given by

$$P = \{\Lambda_j, \mu_P(\Lambda_j); j = 1, 2, ..., n\} \tag{35}$$

where, $\mu_P(\Lambda_j)$ denotes the membership degree satisfying $0 \le \mu_P(\Lambda_j) \le 1$.

Further, let the degree of effectiveness of $\Lambda_j's; (j = 1, 2, ..., n)$ at uniform inputs be denoted by fuzzy set Q given by

$$P = \{\Lambda_j, \mu_Q(\Lambda_j); j = 1, 2, ..., n\} \tag{36}$$

5.1 A New MADM Method

TOPSIS (Technique of Order Preference by Similarity to Ideal Solutions) method is a well-known technique employed for solving multiple attribute decision-making (MADM) problems. In this technique, an option nearest to the positive ideal solution and farthest from the negative ideal solution is chosen as most suitable option. Using the concept of TOPSIS initially introduced by Xu and Yager [24], we now propose a new MADM method to solve the above strategic decision-making problem.Te procedural steps of proposed method are listed as follows:

(1) Determine the best strategy as follows:

$$\tilde{\Lambda}_z = \max(\mu_P(\Lambda_j)); j = 1, 2, ..., n \tag{37}$$

(2) Construct the fuzzy decision matrix as

$$\begin{pmatrix} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_n \\ \Omega_1 & s_{11} & s_{12} & \cdots & s_{1n} \\ \Omega_2 & s_{21} & s_{22} & \cdots & s_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Omega_m & s_{m1} & s_{m2} & \cdots & s_{mn} \end{pmatrix} \quad (38)$$

and $U = [u_1, u_2, \ldots, u_n]$: The number "$(S_{ij})_{1 \le I \le m, 1 \le j \le n}$" in the above matrix denotes the degree of efectiveness of strategy Λ_j corresponding to input Ω_i .

(3) Determine the normalized fuzzy decision matrix $[n_{ij}]$as

$$n_{ij} = \frac{S_{ij}}{\sqrt{\sum_{i=1}^{m} S_{ij}^2}}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n. \quad (39)$$

(4) Determine the weighted normalized fuzzy decision matrix $[\omega_{ij}]$as

$$\omega_{ij} = \mu_i n_{ij}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n. \quad (40)$$

where U = [1, 2, …, 1] denote the weight matrix and μ_j is the weight of jth strategy.

(5) Determine the positive ideal solution (Ω^+) and negative ideal (Ω^-) as

$$\Omega^+ = \{\omega_1^+, \omega_2^+, \ldots., \omega_m^+\}, \quad (41)$$

$$\Omega^- = \{\omega_1^-, \omega_2^-, \ldots., \omega_m^-\}, \quad (42)$$

where $\omega_1^+ = \max(\omega_{ij}), \omega_i^- = \min(\omega_{ij}), i = 1, 2, \ldots, m; j = 1, 2, \ldots, n.$

(6) Determine the separation of each alternative Ω_i(i = 1, 2, … ,m) from Ω^+ and Ω^-, respectively, by using (31). Let these be denoted by $V\hat{G}S(\Omega_i, \Omega^+)$ and $V\hat{G}S(\Omega_i, \Omega)$.

(7) Determine the relative closeness coeffcients of each alternative Ω_i(i = 1, 2, … ,m) from (Ω^+) and (Ω^-) as

$$S_i = \frac{V\hat{G}S(\Omega_i, \Omega^-)}{V\hat{G}S(\Omega_i, \Omega^-) + V\hat{G}S(\Omega_i, \Omega^+)}, i = 1, 2, \ldots, m. \quad (43)$$

(8) Rank the alternatives according to relative closeness coefficients obtained in Step (7). Therefore, if the strategy $\tilde{\Lambda}_z$ z coupled with input Ω_i is applied, then we get optimal output.

An Illustrative Example.

Let us take an example of a person who desires to open a new shopping mall in a city. To know the interest (opinion) of the people of the city and for proper functioning of the shopping mall in terms of sale, the owner has decided to conducted a survey before the inauguration of the mall. Then, the entire assembled perspectives are divided in five different classifications termed as inputs and are denoted by Ω_i; i = 1, 2, 3, 4, 5. On the basis of these collected perspectives, five plans are planned by the owners think tank for the inauguration of the shopping mall in the city represented by Λ_j ; j = 1, 2, 3, 4, 5. Our goal is to obtain a specific plan in such a way that when it is combined with a particular input then we get maximum profits in return. For attaining the complete success of the mall in the city, owner of the mall has decided to make a team of one hundred specialists . The levels of efficacy of various plans at consistent inputs and at specific inputs are shown in tables 11 and 12 respectively. Now, we employ the proposed measure (31) to compute the above example using TOPSIS method defined above.

(1) Applying Step (1) in Table 11, the best strategy so obtained is given by Λ_3.
(2) The calculated fuzzy decision matrix using Step (2) is given in Table 12.
(3) The computed weighted normalized fuzzy decision matrix using (39) and (40) is shown in Table 13.
(4) Computed values of positive ideal solution (Ω^+) and negative ideal solution (Ω^-) using Step (5) are given in Table 14.

Table 11. Efficiencies of Λ^j_s at uniform inputs.

μ_1,(Λ_1)	μ_1,(Λ_2)	μ_1,(Λ_3)	μ_1,(Λ_4)	μ_1,(Λ_5)
.7	.3	.8	.5	.6

(5) Separation measures of Ωis from Ω^+ and Ω^- using (31) are displayed in Table 15.
(6) Calculated values of relative closeness coefficients using Step (7) are presented in

Table 12. Eficiencies of strategies at particular inputs.

Ω_j	μ_Q,(Λ_1)	μ_Q,(Λ_2)	μ_Q,(Λ_3)	μ_Q,(Λ_4)	μ_Q,(Λ_5)
Ω_1	.5	.2	.6	.9	.4
Ω_2	.4	.6	.8	.3	.6
Ω_3	.8	.3	.5	.7	.2
Ω_4	.6	.4	.2	.8	.9
Ω_5	.9	.5	.3	.4	.7

Table 13. Normalized/weighted normalized fuzzy decision matrix

	μ_Q,(Λ_1)	μ_Q,(Λ_2)	μ_Q,(Λ_3)	μ_Q,(Λ_4)	μ_Q,(Λ_5)
Ω_1	.3356	.2108	.5108	.6082	.2933
Ω_2	.2685	.6325	.6810	.2027	.4399
Ω_3	.5369	.3162	.4256	.4730	.1466
Ω_4	.4027	.4216	.1703	.5406	.6599
Ω_5	.6040	.5270	.2554	.2703	.5133

Table 14. Fuzzy positive and negative ideal solutions.

Ω^+	.6082	.6810	.5369	.6599	.6040
Ω^-	.2108	.2027	.146	.1703	.2554

Table 16.

Arranging the alternatives according to values of relative closeness coefficients in descending order, the preferential sequence so obtained is given by the following. Ω_5, Ω_1, Ω_2, Ω_4, Ω_3

Thus the best solution is Ω_5

Now, we compute the above Example by fuzzy MOORA (Multi objective Optimization on the basis of Ratio Analysis) method suggested by Brauers and Zavadskas[36] for the sake of comparison. This method refers to a matrix of responses of alternatives to objectives, to which ratios are applied. The procedural steps are briefed as follows. Fuzzy MOORA Method [36]. Steps (1)-(4) of fuzzy MOORA method are same as Steps (2)-(4) of the proposed method

The remaining steps are as follows:

(5) Compute the overall performance index $V\hat{G}S(\Omega^+, \Omega^-)$ for all $\Omega_i^{'}s, i = 1,2,3,4,5.$ The calculated values are given in Table 7.

(6) Rank the $\Omega_i^{'}s, i = 1,2,3,4,5.$ according to the values of $V\hat{G}S(\Omega^+, \Omega^-)$ obtained in Step (5) in descending order. The ranking results so obtained are shown in Table 17. The ranking order of alternatives is as follows:

$\Omega_5, \Omega_1, \Omega_3, \Omega_4, \Omega_2$

Thus the best alternative, that is, Ω_5, remains unaltered.

Table 15. Distances $\Omega_i s$ from Q^+ and Q^-.

$V\hat{G}S(\Omega_i, \Omega^+)$	$V\hat{G}S(\Omega_i, \Omega^-)$
2.5128	.5713
3.4876	.7327
2.1799	.3444
3.2742	.5596
2.8145	.8902

Table 16. Coefficients of closeness and ranking

Alternative inputs		Ranking
Ω_1	.1852	II
Ω_2	.1736	III
Ω_3	.1364	V
Ω_4	.1459	IV
Ω_4	.2402	I

Table 17. Computed ranks using Moora Method

Alternative inputs		Ranking
Ω_1	.5978	II
Ω_2	.4539	V
Ω_3	.5587	III

Ω_4	.4216	IV
Ω_4	.6903	I

5.2 A comparative Analysis

On comparing the output obtained by using proposed method with that of fuzzy MOORA method proposed by Brauers and Zavadskas [36], we find that the best alternative generated by two methods coincide. But there is a big difference in computational procedures adopted in two methods. In fuzzy MOORA method, the best alternative is decided on the basis of overall performance score values of positive ideal solution and negative ideal solution. In this way, each attribute may not get its due weight-age during the selection of the best alternative, whereas, in proposed method, the best alternative is decided on the basis of relative closeness coefficients; that is, each attribute is given due weight-age in decision-making process [26]. Secondly, in proposed method there is a particular strategy with a specific input whereas this is not the case with fuzzy MOORA method. Thus, the performance of proposed method is considerably good.

6. Conclusion

The two measures (Sine fuzzy entropy and Jensen-Sine fuzzy divergence measure) introduced in this paper are not only found to be the effective alternatives but also the better performing measures than the existing FEMs while handling the problems regarding objective weight computation in MADM problem, ambiguity computation of two different FSs (which do not complement each other) and handling of structured linguistic variables. We successfully applied it to the problems of pattern recognition in contrast with several existing fuzzy similarity measures and distance measures. The performance of the proposed measure is found to be consistent with the existing measures.

7. Availability of data and materials

The datasets supporting the conclusions of this article are included within the article.

8. Conflict of interests

The authors declare that there are no conflict of interests.

References

[1] Zadeh LA. Fuzzy sets. Inf Control.8(3), 338-353 (1965).
[2] Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 20, 87-96(1986).
[3] Shannon CE. A mathematical theory of communication. Bell Syst Tech J. 27, 379-423(1948).
[4] Ebanks BR. On measure of fuzziness and their representation. J Math Anal Appl. 94, 301-312 (1983).
[5] Higashi M, Klir GJ. On measures of fuzziness and fuzzy complements. Int J Gen Syst. 8,169-180 (1982).
[6] Kosko B. Fuzzy entropy and conditioning. Inf Sci. 40,165-174 (1986).

[7] Pal NR, Pal SK. Object background segmentation using new definition of entropy. Proc Inst Electr Eng.136, 284-295(1989).

[8] Pal NR, Pal SK. Higher order fuzzy entropy and hybrid entropy of a set. Inf Sci. 61(3), 211-231 (1992).

[9] Verma R, Sharma BD. On generalized exponential fuzzy entropy. World Acad Sci Eng Technol. 5(12), 1895-1898 (2011).

[10] Joshi R, Kumar S. A new exponential fuzzy entropy of order-(a,) and its application in multiple attribute decision-making problems. Commun Math Stat. 5(2), 213-229(2017).

[11] Kapur JN. Measures of Fuzzy Information. Mathematical Sciences Trust Society; (1997).

[12] Parkash O. A new parametric measure of fuzzy entropy. Inf Process Manage Uncertainty. 2,1732-1737 (1998).

[13] Xuecheng L. Entropy, distance measure and similarity measure of fuzzy sets and their relations. Fuzzy Sets Syst. 52, 305-318(1992).

[14] Chaudhuri BB, Bhattacharya A. On correlation between two fuzzy sets. Fuzzy Sets Syst. 118, 447-456(2001).

[15] Montes S, Couso I, Gil P, Bertoluzza C. Divergence measure between fuzzy sets. Int J Approx Reason. 30, 91-105(2002).

[16] Pappis C, Karacapilidis I. A comparative assessment of measures of similarity of fuzzy values. Fuzzy Sets Syst. 56, 171-174(1993).

[17] Zadeh LA. A fuzzy-set-theoretic interpretation of linguistic hedges. J Cybern. 2, 4-34(1972).

[18] Liu M, Ren H. A new intuitionistic fuzzy entropy and application in multi-attribute decision-making. Inf Sci. 5, 587-601(2014).

[19] De SK, Biswas R, Roy AR. Some operations on intuitionistic fuzzy sets. Fuzzy Sets Syst. 114, 477-484 (2000).

[20] Garg H, Agarwal N, Tripathi A. Generalized intuitionistic fuzzy entropy measure of order a and degree and its applications to multi-criteria decision-making problem. Int J Fuzzy Syst Appl. 6(1), 86-107 (2017).

[21] Chen S, Yeh M, Hsiao P. A comparison of similarity measures of fuzzy values. Fuzzy Sets Syst. 72, 79-89(1995).

[22] Wang WJ. New similarity measures on fuzzy sets and on elements. Fuzzy Sets Syst. 85, 305-309(1997).

[23] Chiang DA, Lin NP. Correlation of fuzzy sets. Fuzzy Sets Syst. 102, 221-226 (1999).

[24] L. A. Zadeh, Fuzzy sets, Information and Control, vol. 8, no. 3, pp. 338353, (1965).

[25] L. A. Zadeh, Probability measures of fuzzy events, Journal of Mathematical Analysis and Applications, vol. 23, pp. 421427, (1968).

[26] A. de Luca and S. Termini, A defnition of a nonprobabilistic entropy in the setting of fuzzy sets theory, Information and Computation, vol. 20, pp. 301312, (1972).

[27] C.M. Hwang and M.-S. Yang, On entropy of fuzzy sets, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, vol. 16, no. 4, pp. 519527, (2008).

[28] R. R. Yager, On the measure of fuzziness and negation. Part I: membership in the unit interval, International Journal of General Systems, vol. 5, no. 4, pp. 221229, (1979).

[29] R. Joshi and S. Kumar, A new exponential fuzzy entropy of order-(a,//) and its application in multiple attribute decision making, Communications in Mathematics and Statistics, vol. 5, no. 2, pp. 213229, (2017).

[30] R. Joshi and S. Kumar, An (R,S)-norm fuzzy relative information measure and its applications in strategic decision making, Computational and Applied Mathematics.

[31] D. Bhandari and N. R. Pal, Some new information measures for fuzzy sets, Information Sciences, vol. 67, no. 3, pp. 209228, (1993).

[32] I. Montes, N. R. Pal, V. Janis, and S. Montes, Divergence Measures for Intuition-istic Fuzzy Sets, IEEE Transactions on Fuzzy Systems, vol. 23, no. 2, pp. 444456, (2015).

[33] R. Joshi and S. Kumar, An (R, S)-norm fuzzy information measure with its applications in multiple-attribute decision-making, Computational and Applied Mathematics.

[34] J. Lin, Divergence measures based on the Shannon entropy, IEEE Transactions on Information Teory, vol. 37, no. 1, pp. 145 151, (1991).

[35] Z. Xu and R. R. Yager, Some geometric aggregation operators based on intu-itionistic fuzzy sets, International Journal of General Systems, vol. 35, no. 4, pp. 417433, (2006).

[36] W. K. M. Brauers and E. K. Zavadskas, The MOORA method and its application to privatization in a transition economy, Control and Cybernetics, vol. 25, no. 2, pp. 445469,

[37] Tsallis, C., Possible Generalization of Boltzman Gibbs Statistics, J. Stat.. Phy., 52, 479-487(1988).

[38] Boekee DE, Vander Lubbe JCA , The R-norm information measure. Inf Control 45, 136-155(1980).